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Supersymmetric AdS_5 black holes

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Abstract: The first examples of supersymmetric, asymptotically AdS_5, black hole solutions are presented. They form a 1-parameter family of solutions of minimal five-dimensional gauged supergravity. Their angular momentum can never vanish. The solutions are obtained by a systematic analysis of supersymmetric solutions with Killing horizons. Other new examples of such solutions are obtained. These include solutions for which the horizon is a homogeneous Nil or $\mathrm{SL}(2,\mathbb{R})$ manifold.

Keywords: Black Holes in String Theory, Supergravity Models.

Contents

1. Introduction

The AdS/CFT correspondence [1] provides a non-perturbative definition of quantum gravity in spacetimes that are asymptotic to a product of anti-de Sitter space with some compact manifold. In principle, it provides a precise framework in which the puzzles of quantum black hole physics should be solvable. The correspondence has certainly led to an improved qualitative understanding of black holes but quantitative results are lacking. For example, it has not been possible to calculate precisely the entropy of, say, a Schwarzschild-AdS black hole. The problem is that we don't know how to compute in strongly coupled gauge theories.

The exception to these remarks is the BTZ black hole [2]. In this case, the Cardy formula of two dimensional CFT can be employed to calculate the black hole entropy [3]. The older string theory calculations of the entropy of asymptotically flat (nearly) supersymmetric black holes [4]–[10] are now understood as arising from this result, since such black holes always arise from black string solutions whose near-horizon geometries are locally products involving an AdS_3 factor [3, 11, 12, 13]. The black hole entropy calculations are applications of AdS_3/CFT_2 to these BTZ-like near-horizon geometries. Hence there is a sense in which the only black holes whose entropy has been calculated are BTZ-like solutions.

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It is clearly desirable to calculate the entropy of higher dimensional AdS black holes using the AdS/CFT correspondence. However, in going to higher dimensions we have to confront the problem of strong coupling in the CFT. One approach that proved succesful in the old entropy calculations was to restrict attention to *supersymmetric* black holes. Non-renormalization theorems then allow certain results to be extrapolated from weak to strong coupling. It is natural to ask whether this can also be done for asymptotically AdS black holes in dimension $D > 3$.

The first obstacle one encounters in attempting to do this is that it is hard to think of any supersymmetric, asymptotically[1] AdS, black hole solutions with $D > 3$. For example, the BPS limit of the Reissner-Nordstrom-AdS solution is a naked singularity, not a black hole [14, 15].

A clue to overcoming this obstacle comes from the BTZ solution. The extremal BTZ solution preserves some supersymmetry [16]. It has mass $M = |J|\ell$ where J is the angular momentum and ℓ the asymptotic AdS radius. However, this solution only describes a black hole when $J \neq 0$, i.e., *supersymmetric* AdS_3 *black holes must rotate.* When $J = 0$ it does not have a regular horizon [17].

The same is true of supersymmetric, asymptotically AdS black holes in four dimensions. If the black hole uniqueness theorems extend to asymptotically AdS spacetimes then the most general stationary black hole solution of $D = 4$ Einstein-Maxwell theory with a negative cosmological constant should belong to the family of charged rotating black hole solutions obtained by Carter [18] (often called Kerr-Newman-AdS solutions). The BPS limit of these solutions was examined in [19] in the context of minimal $\mathcal{N} = 2$ gauged supergravity. It was shown that supersymmetric black holes do indeed exist. Their mass M and angular momentum J are uniquely determined by their charge(s).[2] Once again, there are no supersymmetric black hole solutions with $J = 0$ (BPS black holes with small J also have small M, and reduce to AdS_4 when $J = 0$). Hence all supersymmetric black holes in this theory have non-vanishing angular momentum. This contrasts with the ungauged theory, in which all supersymmetric black holes have vanishing angular momentum.

These supersymmetric black holes solutions can be oxidized to $D = 11$ using results of [21]. This gives supersymmetric, asymptotically $AdS_4 \times S^7$, black hole solutions. One might hope to be able to calculate their entropy by counting BPS operators in the dual CFT [22]. Unfortunately, the dual CFT is only poorly understood and, in particular, the spectrum of BPS operators is not known. We shall therefore turn our attention to five dimensions.

[1] By "asymptotically AdS" we mean a spacetime that is asymptotic to global AdS rather than one which is only asymptotically locally AdS.

[2] There is some controversy over which charges these black holes can carry. In [19], a BPS bound for this theory was presented and it was shown that black holes saturate this bound when M and J take values determined by Q and P, the electric and magnetic charges. This implies that there should be a 2-parameter family of dyonic supersymmetric black holes. However, in [20] it was claimed that existence of a super-covariantly constant spinor implies $P = 0$, i.e., there should only be a 1-parameter family of electrically charged supersymmetric black holes. A similar result for non-rotating (nakedly singular) solutions was obtained in [14]. These results contradict each other since existence of a super-covariantly constant spinor should be equivalent to saturation of the BPS bound.

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The AdS_5/CFT_4 correspondence is better understood than AdS_4/CFT_3 because the CFT has a fairly simple description as $\mathcal{N} = 4$ SU(N) super Yang-Mills theory. A lot is known about BPS operators in this theory. It therefore seems a promising arena in which to study supersymmetric black holes. Unfortunately, there are no known supersymmetric, asymptotically AdS_5, black hole solutions.[3]

The goal of this paper is to obtain the first examples of supersymmetric, asymptotically AdS_5 black holes. Guided by the above discussion of $D = 3, 4$ solutions, we should expect such black holes to have non-zero angular momentum. That is why finding solutions is non-trivial. Fortunately, recent advances in our understanding of supersymmetric supergravity solutions give a systematic method for finding such solutions.

Long ago, Tod obtained all supersymmetric solutions of the minimal $\mathcal{N} = 2$ $D = 4$ ungauged supergravity theory [26]. There has recently been a revival of interest in Tod's work, and his method has now been applied to the minimal supergravity theories in $D = 5$ [27] and $D = 6$ [28] and the minimal gauged supergravity theories in $D = 4$ [29] and $D = 5$ [30] (note that all of these theories have 8 supercharges). The corresponding analysis has also been performed in $D = 11$ [31, 32], although the results become increasingly less explicit as D increases. The results of [27] were used in [33] to prove a uniqueness theorem for supersymmetric black hole solutions of minimal $D = 5$ ungauged supergravity. The proof is constructive, i.e., it yields the black hole solution explicitly. If this proof can be adapted to other theories then it yields a systematic method for obtaining all supersymmetric black hole solutions of such theories.

This strategy was applied to the minimal $D = 6$ ungauged supergravity in [28]. Supersymmetric solutions of this theory are, in general, much more complicated than those of the $D = 5$ theory. This made obtaining a full uniqueness theorem too difficult. Nevertheless, the method of [33] does allow one to determine all possibilities for the near-horizon geometry of a supersymmetric solution with a (spatially) compact event horizon.

In this paper we shall consider the minimal $D = 5$ gauged supergravity theory analyzed in [30]. Applying the method of [33] to this theory reduces finding the near-horizon geometry to solving certain equations on a 3-manifold H corresponding to a spacelike cross-section of the event horizon. We find several families of near-horizon solutions in which H is locally isometric to a homogeneous metric on a group manifold, specifically[4] Nil, SL($2, \mathbb{R}$) or SU(2). In the latter case, the metric is the squashed metric on S^3. There is also the possibility of the near-horizon geometry being AdS_5, in which case H is locally isometric to H^3 with its standard Einstein metric.

The most promising candidate for the near-horizon geometry of a black hole is the solution with $H = S^3$. We therefore use the formalism of [30] to look for supersymmetric solutions that have this near-horizon geometry and are asymptotically AdS_5. We do this by writing both the near-horizon solution and the AdS_5 solution in a canonical form discussed in [30]. The similarity of the two solutions suggests a natural Ansatz for obtaining a

[3]See [23, 24, 25] for some attempts at constructing such solutions. These attempts produced solutions with naked singularities or naked closed timelike curves rather than black hole solutions.

[4]Solutions with H isometric to a homogeneous Nil-manifold have been previously obtained in [34]. Unlike our solutions, these solutions are static and not supersymmetric.

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solution that interpolates between the two. Plugging this Ansatz into the equations of [30] yields a 1-parameter family of asymptotically AdS_5 black hole solutions preserving at least two of the eight supersymmetries.

Our solutions are parametrized by their (electric) charge, which determines their mass and angular momenta. They carry equal angular momentum in two orthogonal 2-planes, just like the supersymmetric black holes of the ungauged $D = 5$ theory [5, 35]. This angular momentum vanishes only when the charge vanishes, when the solution reduces to AdS_5. In this respect they are similar to the $D = 4$ solutions discussed above. Small charge black holes are small, with low angular momentum and almost round horizons whereas large charge black holes are large with high angular momentum and very squashed horizons. The causal structure of our solutions is very similar to that of the black holes of the ungauged theory, which was analyzed in [36]. Behind the horizon there is a naked singularity surrounded by a region of closed timelike curves. Geodesics entering the black hole can emerge into a new asymptotically AdS region.

Our solutions can be oxidized to give BPS solutions of type-IIB supergravity using the results of [21]. We hope that certain properties of these supersymmetric, asymptotically $AdS_5 \times S^5$ black hole solutions will be reliably calculable within the dual $\mathcal{N} = 4$ SU(N) super Yang-Mills theory even at weak coupling.

We should note that, although supersymmetric, asymptotically AdS, black holes in $D = 3, 4, 5$ must rotate, the same is not true in $D = 7$. Supersymmetric, asymptotically AdS_7, black hole solutions were obtained in [37]. These solutions are static. However, they appear to preserve only one supersymmetry, which is probably too few to be useful in attempting to calculate their entropy, especially since the AdS_7/CFT_6 duality is not very well understood.

This paper is arranged as follows. In section 2 we review the general form of supersymmetric solutions of minimal $D = 5$ gauged supergravity, as deduced in [30]. Section 3 contains the analysis of possible near-horizon geometries of supersymmetric solutions with horizons. In section 4 we obtain the supersymmetric black hole solutions. Our results are discussed in section 5.

The reader interested only in the black hole solutions (and not their derivation) should jump to subsection 4.2. Other new supersymmetric solutions are given by equations (3.45), (3.50), (3.57), (4.62) and (4.63).

2. Supersymmetric solutions

The theory we shall be considering is minimal $D = 5$ gauged supergravity, with bosonic action

$$S = \frac{1}{4\pi G} \int \left[\left(\frac{R_5}{4} + \frac{3}{\ell^2} \right) \star 1 - \frac{1}{2} F \wedge \star F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right], \tag{2.1}$$

where R_5 is the Ricci scalar and $F = dA$ is the field strength of the U(1) gauge field. The bosonic equations of motion are

$$^5R_{\alpha\beta} - 2F_{\alpha\gamma}F_\beta{}^\gamma + \frac{1}{3} g_{\alpha\beta} \left(F^2 + \frac{12}{\ell^2} \right) = 0$$

$$d * F + \frac{2}{\sqrt{3}} F \wedge F = 0 \tag{2.2}$$

where $F^2 \equiv F_{\alpha\beta}F^{\alpha\beta}$. The general form of purely bosonic supersymmetric solutions of this theory was obtained in [30]. It was shown that any such solution must admit certain globally defined tensors, namely a real scalar f, a real vector V and three real 2-forms X^i, $i = 1, 2, 3$. f, V and X^1 are gauge-invariant but $X^{2,3}$ are not: $X^2 + iX^3$ picks up a phase under a gauge transformation.

These quantities satisfy certain algebraic relations, which are the same as in the ungauged theory [27]:

$$\begin{aligned}
V^\alpha V_\alpha &= -f^2 \,, && (2.3)\\
X^i \wedge X^j &= -2\delta_{ij} f \star V \,, && (2.4)\\
i_V X^i &= 0 \,, && (2.5)\\
i_V \star X^i &= -f X^i \,, && (2.6)\\
\left(X^i\right)_{\gamma\alpha} \left(X^j\right)^\gamma{}_\beta &= \delta_{ij}\left(f^2 \eta_{\alpha\beta} + V_\alpha V_\beta\right) - \epsilon_{ijk} f \left(X^k\right)_{\alpha\beta} \,, && (2.7)
\end{aligned}$$

where $\epsilon_{123} = +1$ and, for a p-form Ω and vector Y, $i_Y \Omega$ denotes the $(p-1)$-form obtained by contracting Y with the first index of Ω. From the first expression we see that V is timelike or null (the possibility of V vanishing anywhere can be excluded using an argument in [33]).

There are also differential relations [30]:

$$\begin{aligned}
df &= -\frac{2}{\sqrt{3}} i_V F \,, && (2.8)\\
D_{(\alpha} V_{\beta)} &= 0 \,, && (2.9)\\
dV &= -\frac{4}{\sqrt{3}} f F - \frac{2}{\sqrt{3}} \star (F \wedge V) - 2\ell^{-1} X^1 && (2.10)
\end{aligned}$$

and

$$dX^i = \frac{1}{\ell} \epsilon_{1ij} \left[2\sqrt{3} A \wedge X^j + 3 \star X^j\right] , \tag{2.11}$$

hence X^1 is closed, but $X^{2,3}$ need not be. Equations (2.9) and (2.8) imply that V is a Killing vector field that leaves the Maxwell field strength invariant [30], i.e., V generates a symmetry of the solution.

There are two cases to consider. In the first case, V is globally null, i.e., f vanishes everywhere. A general analysis of such solutions was presented in [30] where it was shown that they preserve at least 1/4 of the supersymmetry. We shall refer to these as the null family of solutions.

We shall be primarily interested in the case in which V is not globally null. We shall refer to such solutions as belonging to the timelike family because for these solutions we can always find some open set $\mathcal{U}$ in which V is timelike. There is no loss of generality in assuming $f > 0$ in $\mathcal{U}$. Coordinates can be introduced so that the metric in $\mathcal{U}$ takes the form

$$ds^2 = -f^2(dt + \omega)^2 + f^{-1} ds_4^2 \,, \tag{2.12}$$

where $V = \partial/\partial t$, ds_4^2 is the line element of a four-dimensional riemannian "base space" $\mathcal{B}$ orthogonal to the orbits of V, and ω is a 1-form on $\mathcal{B}$. The strategy of [30] (following [27]) is to reduce necessary and sufficient conditions for supersymmetry to a set of equations on $\mathcal{B}$.

Equation (2.5) implies that the 2-forms X^i can be regarded as 2-forms on $\mathcal{B}$, and equation (2.6) implies that these are anti-self dual if we take the volume form η_4 of $\mathcal{B}$ to be related to the five-dimensional volume form η by

$$\eta_4 = f i_V \eta \,. \tag{2.13}$$

Equation (2.7) implies that the 2-forms obey the algebra of the imaginary unit quaternions on $\mathcal{B}$, i.e., they define an almost hyper-Kähler structure. This is not integrable: supersymmetry merely requires that $\mathcal{B}$ is Kähler, with Kähler form X^1 [30].

Equations (2.8) and (2.10) can be solved to determine the field strength, giving [30]

$$F = \frac{\sqrt{3}}{2} d\left[f(dt+\omega)\right] - \frac{1}{\sqrt{3}} G^+ - \sqrt{3}\ell^{-1} f^{-1} X^1 \,, \tag{2.14}$$

where $G^\pm$ is defined by

$$G^\pm = \frac{f}{2}\left(d\omega \pm \star_4 d\omega\right) , \tag{2.15}$$

and $\star_4$ denotes the Hodge dual on $\mathcal{B}$. Supersymmetry requires that [30]

$$f^{-1} = -\frac{\ell^2 R}{24} \,, \tag{2.16}$$

where R is the Ricci scalar of $\mathcal{B}$, and

$$G^+ = -\frac{\ell}{2}\left(\mathcal{R} - \frac{R}{4} X^1\right) \tag{2.17}$$

where $\mathcal{R}$ is the Ricci form on $\mathcal{B}$, defined by ($m, n, \ldots$ denote curved indices on $\mathcal{B}$)

$$\mathcal{R}_{mn} = \frac{1}{2} R_{mnpq} \left(X^1\right)^{pq} \,, \tag{2.18}$$

with R_{mnpq} the Riemann tensor of $\mathcal{B}$.

The final condition arises from the Maxwell equation, which gives [30]

$$\nabla^2 f^{-1} = \frac{2}{9}(G^+)^{mn}(G^+)_{mn} + \ell^{-1} f^{-1} (G^-)^{mn} (X^1)_{mn} - 8\ell^{-2} f^{-2} \,, \tag{2.19}$$

where ∇^2 is the laplacian on $\mathcal{B}$. These conditions are necessary for supersymmetry. It turns out that they are also sufficient, with the solution preserving at least 1/4 of the supersymmetry [30]. Hence all supersymmetric solutions in the timelike family are determined as follows. First pick a Kähler manifold $\mathcal{B}$ with negative Ricci scalar. Equation (2.16) determines f. Now find a 1-form ω so that equations (2.17) and (2.19) are satisfied (where X^1 is the Kähler form). It was demonstrated in [30] that a solution always exists. Then there will be some region $\mathcal{U}$ of spacetime in which the metric is given by (2.12) and the field strength by (2.14), which can also be written

$$F = \frac{\sqrt{3}}{2} d\left[f(dt + \omega)\right] + \frac{\ell}{2\sqrt{3}} \mathcal{R} \,. \tag{2.20}$$

The solution outside of $\mathcal{U}$ can then be obtained by analytic continuation.

Note that the only information used in deriving the general supersymmetric solution (in either family) is the equations satisfied by f, V and X^i. Hence these equations are both necessary and sufficient for supersymmetry.

We should mention that some solutions belong to both the timelike and null families. This is only possible for solutions preserving more than 1/4 of the supersymmetry. Examples are the maximally supersymmetric AdS_5 solution, and the $AdS_3 \times H^2$ solution [30]. Since AdS_5 is the unique maximally supersymmetric solution [30], the latter solution presumably preserves 1/2 or 3/4 of the supersymmetry.

3. Near horizon analysis

3.1 General analysis

If a supersymmetric solution has a physical horizon $\mathcal{H}$ then it must be preserved by all Killing vectors of the spacetime and in particular by the supersymmetric Killing vector field V. This implies that V must be spacelike or null on the horizon. However, we know that V cannot be spacelike hence V must be null on the horizon, i.e., the horizon is a Killing horizon of V.

If a timelike solution has a horizon then $\mathcal{H}$ cannot intersect $\mathcal{U}$ because V is null on $\mathcal{H}$ but timelike in $\mathcal{U}$. Therefore it is not possible to identify which solutions have horizons just by looking at the solution in the form presented above. We shall instead adopt the approach introduced in [33]. This approach is equally valid for timelike or null solutions.

The idea is to introduce a coordinate system adapted to the presence of a Killing horizon, namely gaussian null coordinates, and then to examine the equations satisfied by f, V and X^i in such coordinates (recall that these are necessary and sufficient for supersymmetry). In the near-horizon limit, they reduce to equations on a 3-manifold H corresponding to a constant time slice through $\mathcal{H}$ (i.e. $H = \mathcal{H} \cap \Sigma$ where Σ is a spacelike slice that intersects $\mathcal{H}$). In the ungauged theory considered in [33] it was shown that these equations can be completely solved when H is compact and the near-horizon geometry can therefore be determined explicitly. This has also been done in the minimal six-dimensional ungauged supergravity [28].

In gaussian null coordinates, the line element takes the form [33]

$$ds^2 = -r^2\Delta^2 du^2 + 2dudr + 2rh_A du dx^A + \gamma_{AB} dx^A dx^B \tag{3.1}$$

with

$$V = \frac{\partial}{\partial u}, \qquad f = r\Delta \,. \tag{3.2}$$

The horizon is at $r = 0$, and H is given by $r = 0$ and $u =$ constant. The region exterior to the horizon is $r > 0$, where we can assume $\Delta \geq 0$. The quantities Δ, h_A and γ_{AB} depend smoothly on r and x^A but are independent of u (because V is Killing).

The near-horizon limit is defined by $r = \epsilon\tilde{r}$ and $u = \tilde{u}/\epsilon$ with $\epsilon \to 0$. After taking this limit we recover a metric of the same form but with Δ, h_A and γ_{AB} depending only on x^A. This is the reason why obtaining the near-horizon geometry of a supersymmetric solution reduces to solving equations on H. In what follows, we shall *not* assume that the near-horizon limit has been taken since it is not immediately obvious that this limit must preserve supersymmetry. We shall instead evaluate all equations as a power series in r. The near-horizon limit corresponds to discarding all but the $\mathcal{O}(r^0)$ terms.

We are free to choose a gauge in which the gauge field A is a smooth function of r with

$$A_u \equiv i_V A = \frac{\sqrt{3}}{2} f \,. \tag{3.3}$$

In this gauge, we have $\mathcal{L}_V A = 0$ (using equation (2.8)). Write

$$A = \frac{\sqrt{3}}{2} r\Delta du + A_r dr + a_A dx^A \,. \tag{3.4}$$

In general, a_A will not be globally defined on H. Note also that taking the near-horizon limit removes A_r.

The 2-forms X^i can be written as [33]

$$X^i = dr \wedge Z^i + r\left(h \wedge Z^i - \Delta \star_3 Z^i\right), \tag{3.5}$$

where $h \equiv h_A dx^A$, $\star_3$ denotes the Hodge dual with respect to γ_{AB} (the volume form η_3 of γ_{AB} is chosen so that $du \wedge dr \wedge \eta_3$ has positive five-dimensional orientation), and

$$Z^i = Z^i_A dx^A \tag{3.6}$$

are a set of vector fields orthonormal with respect to γ_{AB}, i.e.,

$$\star_3 Z^i = \frac{1}{2}\epsilon_{ijk} Z^j \wedge Z^k \,. \tag{3.7}$$

Z^1 is gauge-invariant and hence globally defined on H. However, $Z^{2,3}$ are gauge-dependent: if we define

$$W \equiv Z^2 + iZ^3 \tag{3.8}$$

then W picks up a phase under a gauge transformation. Equation (2.11) gives

$$\begin{aligned}\hat{d}Z^i &= h \wedge Z^i - \Delta \star_3 Z^i + r\partial_r(h \wedge Z^i - \Delta \star_3 Z^i) + \\ &\quad + \ell^{-1}\epsilon_{1ij}\left[3 \star_3 Z^j + 2\sqrt{3} a \wedge Z^j - 2\sqrt{3} r A_r \left(h \wedge Z^j - \Delta \star_3 Z^j\right)\right],\end{aligned} \tag{3.9}$$

and also

$$\begin{aligned}\star_3 \hat{d}h &= \hat{d}\Delta + \Delta h - r(\partial_r\Delta)h + 2r\Delta\partial_r h + r \star_3 (h \wedge \partial_r h) + r\Delta^2 \epsilon_{ijk} Z^i \langle Z^j, \partial_r Z^k\rangle + \\ &\quad + \frac{\Delta}{\ell}\left(6 + 4\sqrt{3} r A_r \Delta\right) Z^1 \,.\end{aligned} \tag{3.10}$$

In these equations, $\langle , \rangle$ denotes the inner product defined by γ_{AB}, and $\hat{d}$ is defined by

$$(\hat{d}Y)_{ABC\ldots} = (p+1)\partial_{[A}Y_{BC\ldots]} \tag{3.11}$$

for any p-form Y with only $A, B, C, \ldots$ indices. To leading order in r, these equations reduce to

$$\hat{d}Z^i = -\Delta \star_3 Z^i + h \wedge Z^i + \ell^{-1}\epsilon_{1ij}\left(3\star_3 Z^j + 2\sqrt{3}a \wedge Z^j\right) + \mathcal{O}(r) \tag{3.12}$$

and

$$\star_3 \hat{d}h - \hat{d}\Delta - \Delta h = \frac{6\Delta}{\ell}Z^1 + \mathcal{O}(r)\,. \tag{3.13}$$

We shall be interested in evaluating such equations at $r = 0$, i.e., on the 3-manifold H. From equation (3.7) we obtain

$$\hat{d}^\dagger Z^i = 2\left(h \cdot Z^i + 3\ell^{-1}\delta_{i1}\right) + 2\sqrt{3}\ell^{-1}\epsilon_{1ij}a \cdot Z^j + \mathcal{O}(r)\,, \tag{3.14}$$

where $\hat{d}^\dagger \equiv \star_3\hat{d}\star_3$, and $h \cdot Z^i$ and $a \cdot Z^j$ are defined by contracting indices with γ^{AB}.

For $r > 0$, if $\Delta > 0$ then V is timelike and hence G^+ is well-defined. Self-duality implies that G^+ can be written [33]

$$G^+ = dr \wedge \mathcal{G} + r\left(h \wedge \mathcal{G} + \Delta \star_3 \mathcal{G}\right), \tag{3.15}$$

where $\mathcal{G} \equiv \mathcal{G}_A dx^A$. We obtain

$$\begin{aligned}\mathcal{G} \;=\; & -\frac{3}{2r\Delta^2}\hat{d}\Delta + \frac{3}{2\Delta^2}(\partial_r\Delta)h - \frac{3}{2\Delta}\partial_r h - \frac{1}{2}\epsilon_{ijk}Z^i\langle Z^j, \partial_r Z^k\rangle - \\ & - \frac{1}{2r\ell\Delta}\left(6 + 4\sqrt{3}rA_r\Delta\right)Z^1\,,\end{aligned} \tag{3.16}$$

where we have used equation (3.10) to eliminate $\hat{d}h$. We can now determine the Maxwell field strength from equation (2.14):

$$\begin{aligned}F \;=\; & \frac{\sqrt{3}}{2}\Bigg[-\partial_r(r\Delta)du \wedge dr - r du \wedge \hat{d}\Delta + \frac{1}{3}\epsilon_{ijk}dr \wedge Z^i\langle Z^j, \partial_r Z^k\rangle - \\ & - \star_3 h - r\star_3\partial_r h + \frac{r}{3}(-2\Delta\star_3 Z^i + h \wedge Z^i)\langle Z^j, \partial_r Z^k\rangle\Bigg] + \\ & + \ell^{-1}\left[2A_r\left(dr \wedge Z^1 + rh \wedge Z^1\right) - \left(\sqrt{3} + 4rA_r\Delta\right)\star_3 Z^1\right].\end{aligned} \tag{3.17}$$

Once again, we have used equation (3.10) to eliminate $\hat{d}h$. Note that F is regular at $r = 0$ even if $\Delta = 0$. Despite the appearance of the gauge-dependent quantity A_r, this expression is gauge invariant because there are terms involving $Z^{2,3}$, which are also gauge-dependent. Under a gauge transformation, the gauge dependence of these different quantities must cancel. In any case, we shall only be interested in evaluating F at $r = 0$ where gauge-independence is manifest. For example, the AB component of F is given by

$$\frac{1}{2}F_{AB}dx^A \wedge dx^B = -\frac{\sqrt{3}}{2}\star_3\left(h + 2\ell^{-1}Z^1\right) + \mathcal{O}(r)\,, \tag{3.18}$$

hence

$$\hat{d}a = -\frac{\sqrt{3}}{2} \star_3 \left(h + 2\ell^{-1} Z^1\right) + \mathcal{O}(r)\,. \tag{3.19}$$

The ABC component of the Bianchi identity for F is therefore

$$\hat{d}\left[\star_3 \left(h + 2\ell^{-1} Z^1\right)\right] = \mathcal{O}(r)\,. \tag{3.20}$$

Combining this with equation (3.14) gives

$$\hat{d}^\dagger h = -4\ell^{-1}\left(h \cdot Z^1 + 3\ell^{-1}\right) + \mathcal{O}(r)\,. \tag{3.21}$$

Note that this implies that h cannot be identically zero on H. The next step of [33] is to consider

$$\hat{d}\Delta \wedge \star_3 \hat{d}\Delta = \hat{d}\left(\Delta \hat{d}h - \frac{\Delta^2}{2}\star_3 h - 3\ell^{-1}\Delta^2 \star_3 Z^1\right) + 4\ell^{-1}\Delta^2\left(h \cdot Z^1 + 3\ell^{-1}\right) + \mathcal{O}(r)\,. \tag{3.22}$$

where we have used equations (3.13), (3.14) and (3.21). The expression inside the bracket is globally defined on H. Hence in the ungauged theory ($\ell = \infty$) the right hand side is exact, so integrating this expression over H implies that Δ is constant on H if H is compact. However this simple argument is no longer possible in the gauged theory owing to the presence of the final term above.

We can calculate the spin connection using equation (3.12). This gives

$$\begin{aligned}\nabla_A Z^i_B = & -\frac{\Delta}{2}\left(\star_3 Z^i\right)_{AB} + \gamma_{AB}\left(h \cdot Z^i + 3\ell^{-1}\delta_{i1}\right) - Z^i_A h_B - 3\ell^{-1} Z^i_A Z^1_B + \\ & + 2\sqrt{3}\ell^{-1}\epsilon_{1ij} a_A Z^j_B + \mathcal{O}(r),\end{aligned} \tag{3.23}$$

where ∇ is the connection associated with γ_{AB}. The Ricci tensor R_{AB} of γ_{AB} can now be obtained using

$$R_{AB} Z^{iB} = \nabla_B \nabla_A Z^{iB} - \nabla_A \nabla_B Z^{iB}\,. \tag{3.24}$$

All gauge dependent terms cancel, as they must, giving

$$R_{AB} = \left(\frac{\Delta^2}{2} + h \cdot h + 4\ell^{-1} h \cdot Z^1\right)\gamma_{AB} - h_A h_B - \nabla_{(A} h_{B)} - 6\ell^{-1} h_{(A} Z^1_{B)} - 6\ell^{-2} Z^1_A Z^1_B + \mathcal{O}(r)\,. \tag{3.25}$$

So far we have not assumed that H is compact. In the ungauged theory, it is possible to exploit compactness of H to prove that h must be a Killing vector field on H [33]. In the gauged theory, it is natural to guess that if such a Killing vector field exists then it will be a linear combination of h and Z^1. To look for such a Killing vector field, it is natural to define

$$N = h + 2\ell^{-1} Z^1\,, \tag{3.26}$$

which is coclosed on H. Following the strategy of [33], we look for a value of α such that $N + \alpha Z^1$ is Killing. To do this, define

$$I \equiv \int_H \nabla_{(A}(N + \alpha Z^1)_{B)} \nabla^{(A}(N + \alpha Z^1)^{B)}\,. \tag{3.27}$$

We need to find a value for α which gives $I = 0$. I can be calculated by integration by parts, and commuting derivatives using the above expression for R_{AB}. We find

$$\int_H \nabla_{(A}N_{B)}\nabla^{(A}N^{B)} = \int_H \left[\frac{2}{\ell}N\cdot Z^1 N\cdot N + \frac{4}{\ell^2}N\cdot N - \frac{2}{\ell^2}(N\cdot Z^1)^2\right], \tag{3.28}$$

$$\begin{aligned}\int_H \nabla_{(A}Z^1_{B)}\nabla^{(A}N^{B)} &= \int_H \left[\frac{2}{\ell}N\cdot N - \frac{1}{2\ell}(N\cdot Z^1)^2 - \frac{2}{\ell^3} - \right.\\ &\qquad \left. - \frac{1}{2}N\cdot Z^1 N\cdot N - \Delta^2\left(N\cdot Z^1 + \frac{1}{\ell}\right)\right]\\ &= \int_H \left[\frac{1}{2}(N\cdot N)(N\cdot Z^1) + \frac{5}{2\ell}(N\cdot Z^1)^2 - \frac{2}{\ell^3}\right]\end{aligned} \tag{3.29}$$

$$\int_H \nabla_{(A}Z^1_{B)}\nabla^{(A}Z^{1B)} = \int_H \left[\frac{1}{2}N\cdot N + \frac{3}{2}(N\cdot Z^1)^2 - \frac{2}{\ell^2}\right]. \tag{3.30}$$

The first expression in (3.29) is obtained by using integration by parts to remove the derivative on N. The second expression is obtained by substituting in equation (3.23) and then integrating some terms by parts.

Note that (3.29) together with (3.22) implies that

$$\int_H \hat{d}\Delta\wedge\star_3\hat{d}\Delta = 4\ell^{-1}\int_H \Delta^2((N\cdot Z^1)+\ell^{-1}) = 4\ell^{-1}\int_H \frac{2}{\ell}N\cdot N - \frac{3}{\ell}(N\cdot Z^1)^2 - N\cdot Z^1 N\cdot N\,. \tag{3.31}$$

Unfortunately, using only the above expressions for the integrals, there is no value for α that guarantees $I = 0$. In the ungauged theories considered in [33] and [28] it was shown that, when H is compact, Δ must be constant on H and that H must admit a Killing vector. We have been unable to establish the same results in the gauged theory. This prevents us from classifying all possible near-horizon solutions. Nevertheless, we shall now see that it is still posssible to use the above results to obtain particular examples of near-horizon solutions.

3.2 Near-horizon solutions

In this subsection, we shall be interested in the near-horizon solution. As mentioned above, this amounts to neglecting the $\mathcal{O}(r)$ terms, which we shall do henceforth. Determining the near-horizon geometry is equivalent to finding a solution (Z^i,Δ,h,a) of equations (3.7), (3.12), (3.13) and (3.19) on H. The other equations are consequences of these ones. Unfortunately, we have not been able to find the general solution of these equations even for compact H. We shall resort to additional assumptions to obtain solutions.

Guided by the results of [33], we shall first assume that Δ is constant and non-zero on H. Equation (3.13) then implies

$$\hat{d}^\dagger\left(h + 6\ell^{-1}Z^1\right) = 0\,. \tag{3.32}$$

Combining this with equations (3.14) and (3.21) implies

$$h\cdot Z^1 = -\frac{3}{\ell}, \qquad N\cdot Z^1 = -\frac{1}{\ell}\,. \tag{3.33}$$

Assuming H is compact, substituting $N \cdot Z^1 = -\ell^{-1}$ into (3.31) we find that

$$\int_H N \cdot N - (N \cdot Z^1)^2 = 0 \tag{3.34}$$

and hence

$$h = -\frac{3}{\ell} Z^1 \,. \tag{3.35}$$

Hence from (3.23) it follows that Z^1 is Killing, and hence h is Killing.

Conversely, instead of assuming that Δ is constant, we could assume that h is Killing on H, and H is compact. Then $\hat{d}^\dagger h = 0$, so from (3.21) we find $N \cdot Z^1 = -\ell^{-1}$ and hence from (3.31) we can conclude that Δ is constant. Equation (3.35) then follows as above if $\Delta \neq 0$, and it is also easily derived if $\Delta = 0$.

Another starting point would be to assume that $h = \alpha Z^1$ for some constant α. Equations (3.21) and (3.14) then imply $\alpha = -3/\ell$ or $\alpha = -2/\ell$. In the former case, it follows from equation 3.23 that Z^1 and h are Killing, which is the case just discussed. At the end of this subsection, we shall show that the latter case gives an AdS_5 near-horizon geometry.

Motivated by these results, we shall now construct the near-horizon geometry of any supersymmetric solution that has Δ constant on H, and h, Z^1 Killing on H with $h = -(3/\ell)Z^1$. We shall not assume that H is compact or that Δ is non-zero on H.

Equation (3.12) implies (without any assumptions)

$$dW = \left[(-i\Delta + 3\ell^{-1})Z^1 + h - 2\sqrt{3}i\ell^{-1}a\right] \wedge W \,, \tag{3.36}$$

hence $W \wedge dW = 0$ on H so locally we can write

$$W = \sqrt{2}\lambda dw \,, \tag{3.37}$$

where λ and w are locally defined complex functions on H. w is gauge-invariant but λ is not. Locally, we can always perform a gauge transformation to make λ real, which we shall assume henceforth.

We can introduce real functions x, y by writing $w = 2^{-1/2}(x + iy)$. Note that Z^1 is orthogonal to dx and dy so the integral curves of Z^1 lie within surfaces of constant x and y. Hence we can define coordinates (x, y, z) where z is the parameter along the integral curves of Z^1, i.e., $Z^1 = \partial/\partial z$. The metric on H must take the form

$$ds_3^2 = (dz + \alpha)^2 + 2\lambda^2 dw d\bar{w} \,, \tag{3.38}$$

where $\alpha = \alpha_w dw + \alpha_{\bar{w}} d\bar{w}$. The assumption that Z^1 is Killing implies that α and λ are independent of z. We also have

$$h = -3\ell^{-1}(dz + \alpha) \,. \tag{3.39}$$

Equation (3.12) with $h \propto Z^1$ becomes

$$\partial_w \alpha_{\bar{w}} - \partial_{\bar{w}} \alpha_w = -i\Delta\lambda^2 \,. \tag{3.40}$$

This equation determines α up to a gradient which can be absorbed into the definition of z. Equation (3.36) can be solved for a, giving

$$a = -\frac{\Delta \ell}{2\sqrt{3}}(dz + \alpha) - \frac{i\ell}{2\sqrt{3}}\left(\frac{\partial_w \lambda}{\lambda}dw - \frac{\partial_{\bar{w}}\lambda}{\lambda}d\bar{w}\right). \tag{3.41}$$

Substituting this into equation (3.19) gives

$$\partial_w \partial_{\bar{w}} \log \lambda = \frac{1}{2}\left(3\ell^{-2} - \Delta^2\right)\lambda^2 . \tag{3.42}$$

This is Liouville's equation. There are three cases to consider.

The first case is $\Delta = \sqrt{3}/\ell$. We then have $\log \lambda = \mathcal{F}(w) + \bar{\mathcal{F}}(\bar{w})$ where $\mathcal{F}$ is an arbitrary holomorphic function. A holomorphic change of coordinate $w \to w'(w)$ can be used to set $\mathcal{F} = 0$, i.e., $\lambda = 1$. Solving for α gives

$$\alpha = \frac{\sqrt{3}}{2\ell}(ydx - xdy) , \tag{3.43}$$

hence the metric of H is

$$ds_3^2 = \left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)^2 + dx^2 + dy^2 , \tag{3.44}$$

the standard homogeneous metric on the Nil group manifold. The near-horizon limit of such a solution is

$$\begin{aligned} ds^2 &= -\frac{3r^2}{\ell^2}du^2 + 2dudr - \frac{6r}{\ell}du\left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right) + \\ &\quad + \left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)^2 + dx^2 + dy^2 \\ F &= -\frac{3}{2\ell}du \wedge dr + \frac{\sqrt{3}}{2\ell}dx \wedge dy . \end{aligned} \tag{3.45}$$

Note that dimensional reduction on $\partial/\partial z$ gives an $AdS_2 \times R^2$ geometry.

The second case is $0 \leq \Delta < \sqrt{3}/\ell$. The general solution to the Liouville equation is

$$\lambda^2 = \frac{2\mathcal{F}'(w)\bar{\mathcal{F}}'(\bar{w})}{(\mathcal{F}(w) + \bar{\mathcal{F}}(\bar{w}))^2(3\ell^{-2} - \Delta^2)} , \tag{3.46}$$

where $\mathcal{F}$ is an arbitrary holomorphic function. A holomorphic change of coordinates $w \to w'(w)$ can be used to set $\mathcal{F} = w$, i.e.,

$$\lambda^2 = \frac{1}{(3\ell^{-2} - \Delta^2)x^2} . \tag{3.47}$$

Solving for α gives

$$\alpha = \frac{\Delta}{(3\ell^{-2} - \Delta^2)}\frac{dy}{x} , \tag{3.48}$$

hence the metric of H is

$$ds_3^2 = \left(dz + \frac{\Delta}{(3\ell^{-2}-\Delta^2)}\frac{dy}{x}\right)^2 + \frac{1}{3\ell^{-2}-\Delta^2}\left(\frac{dx^2+dy^2}{x^2}\right), \tag{3.49}$$

the standard homogeneous riemannian metric on the $SL(2,\mathbb{R})$ group manifold, except for $\Delta = 0$ when it degenerates to $R \times H^2$. Taking the near-horizon limit gives the solution

$$\begin{aligned} ds^2 &= -r^2\Delta^2 du^2 + 2dudr - \frac{6r}{\ell}du\left(dz + \frac{\Delta}{(3\ell^{-2}-\Delta^2)}\frac{dy}{x}\right) + \\ &\quad + \left(dz + \frac{\Delta}{(3\ell^{-2}-\Delta^2)}\frac{dy}{x}\right)^2 + \frac{1}{3\ell^{-2}-\Delta^2}\left(\frac{dx^2+dy^2}{x^2}\right) \\ F &= -\frac{\sqrt{3}}{2}\Delta du\wedge dr + \frac{\sqrt{3}}{2\ell(3\ell^{-2}-\Delta^2)x^2}dx\wedge dy\,, \end{aligned} \tag{3.50}$$

where Δ is constant everywhere. Dimensional reduction on $\partial/\partial z$ yields an $AdS_2 \times H^2$ geometry. If $\Delta = 0$ then the five dimensional geometry is $AdS_3 \times H^2$, the near-horizon geometry of the black strings of [38].

The third case is $\Delta > \sqrt{3}/\ell$. Solving the Liouville equation gives

$$\lambda^2 = \frac{2\mathcal{F}'(w)\bar{\mathcal{F}}'(\bar{w})}{(1+\mathcal{F}(w)\bar{\mathcal{F}}(\bar{w}))^2(\Delta^2-3\ell^{-2})}\,, \tag{3.51}$$

where $\mathcal{F}$ is an arbitrary holomorphic function. A holomorphic change of coordinate can be used to set $\mathcal{F} = w$. It is then convenient to introduce real coordinates θ, ϕ defined by

$$w = \tan\left(\frac{\theta}{2}\right)e^{i\psi}\,, \tag{3.52}$$

giving

$$\lambda^2 = \frac{2\cos^4(\theta/2)}{\Delta^2-3\ell^{-2}}\,, \tag{3.53}$$

$$\alpha = \frac{\Delta}{\Delta^2-3\ell^{-2}}\cos\theta d\psi\,, \tag{3.54}$$

and the metric on H is

$$ds_3^2 = \frac{1}{\Delta^2-3\ell^{-2}}\left[\frac{\Delta^2}{\Delta^2-3\ell^{-2}}\left(d\phi''+\cos\theta d\psi\right)^2 + d\theta^2 + \sin^2\theta d\psi^2\right], \tag{3.55}$$

where ϕ'' is defined by

$$z = \frac{\Delta}{\Delta^2-3\ell^{-2}}\phi''\,. \tag{3.56}$$

We have included the primes to avoid confusion with other coordinates ϕ, ϕ' to be introduced later. We see that H has the homogeneous geometry of a squashed S^3, i.e. the $SU(2)$ group manifold. Large Δ corresponds to a small almost round S^3 whereas Δ close to $\sqrt{3}/\ell$ corresponds to a large, highly squashed S^3. The near horizon solution is

$$\begin{aligned} ds^2 &= -r^2\Delta^2du^2 + 2dudr - \frac{6\Delta r}{\ell(\Delta^2-3\ell^{-2})}du(d\phi''+\cos\theta d\psi) + \\ &\quad + \frac{1}{\Delta^2-3\ell^{-2}}\left[\frac{\Delta^2}{\Delta^2-3\ell^{-2}}\left(d\phi''+\cos\theta d\psi\right)^2 + d\theta^2 + \sin^2\theta d\psi^2\right] \\ F &= -\frac{\sqrt{3}}{2}\Delta du\wedge dr + \frac{\sqrt{3}\sin\theta}{2\ell(\Delta^2-3\ell^{-2})}d\theta\wedge d\psi\,, \end{aligned} \tag{3.57}$$

where Δ is constant everywhere. Dimensional reduction on $\partial/\partial\phi''$ yields an $AdS_2 \times S^2$ geometry.

In each of these cases, the metric on H is that of a homogeneous metric on a group manifold. This is a purely local result — the global topology of H may differ from that of the group manifold by discrete identifications if this can be done consistently with supersymmetry.

Finally, we mentioned above the possibility of solutions with $h = -(2/\ell)Z^1$ on H. In this case, it is easy to see that equations (3.12) and (3.13) imply $\Delta = 0$ and $dZ^1 = 0$ on H. Hence there is some locally defined function z on H such that $Z^1 = dz$. Equation (3.19) shows that we can choose the gauge $a = 0$ on H, then (3.36) implies $\exp(-z/\ell)W$ is closed on H so locally there is some complex function w on H such that $W = \sqrt{2}\exp(z/\ell)dw$. It follows that the metric on H is

$$ds_3^2 = dz^2 + 2e^{2z/\ell}dwd\bar{w}\,, \tag{3.58}$$

the standard Einstein metric on H^3. Note that this is an example of a solution for which h and Z^1 are not Killing. Taking the near-horizon limit of such a solution gives the solution

$$\begin{aligned} ds^2 &= 2dudr - \frac{4r}{\ell}dudz + dz^2 + 2e^{2z/\ell}dwd\bar{w} \\ F &= 0\,, \end{aligned} \tag{3.59}$$

which is just AdS_5 with vanishing electromagnetic field. Written in this form, the solution belongs to the null family.

3.3 The base space

If we view the above near-horizon geometries as solutions in their own right then, for $r > 0$, V is timelike and the solutions belong to the timelike class of [30] (with the exception of the solutions with $\Delta = 0$, which are in the null class) and are therefore supersymmetric. In general, the base space of the metric (3.1) has metric

$$ds_4^2 = r\Delta\gamma_{AB}dx^A dx^B + \frac{r}{\Delta}\left(\frac{dr}{r} + h_A dx^A\right)^2. \tag{3.60}$$

For the above solutions, this is

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}\left[\Delta^{-2}\left(\Delta^2 + 9\ell^{-2}\right)^2\left(dz' + \alpha\right)^2 + \left(\Delta^2 + 9\ell^{-2}\right)2\lambda^2 dwd\bar{w}\right], \tag{3.61}$$

where we have defined new coordinates z' and ρ by

$$z' = z - \frac{3\log(r/\ell)}{\ell\left(\Delta^2 + 9\ell^{-2}\right)}, \qquad \rho = 2\left(\frac{\Delta r}{\Delta^2 + 9\ell^{-2}}\right)^{1/2}. \tag{3.62}$$

The Kähler form is

$$X^1 = \frac{1}{4}\Delta^{-1}\left(\Delta^2 + 9\ell^{-2}\right)d\left[\rho^2\left(dz' + \alpha\right)\right]. \tag{3.63}$$

Explicitly, the base space metric for the solution (3.45) ($\Delta = \sqrt{3}/\ell$) is

$$ds_4^2 = d\rho^2 + \frac{12\rho^2}{\ell^2}\left[dz' + \frac{\sqrt{3}}{2\ell}\left(ydx - xdy\right)\right]^2 + \frac{3\rho^2}{\ell^2}\left(dx^2 + dy^2\right), \tag{3.64}$$

for the solution (3.50) (with $0 < \Delta < \sqrt{3}/\ell$) it is

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}\left[\Delta^{-2}\left(9\ell^{-2} + \Delta^2\right)^2\left(dz' + \frac{\Delta dy}{\left(3\ell^{-2} - \Delta^2\right)x}\right)^2 + \left(\frac{9\ell^{-2} + \Delta^2}{3\ell^{-2} - \Delta^2}\right)\frac{dx^2 + dy^2}{x^2}\right], \tag{3.65}$$

and for the solution (3.57) ($\Delta > \sqrt{3}/\ell$) it is

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}\left[\left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right)^2\left(d\phi + \cos\theta d\psi\right)^2 + \left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \tag{3.66}$$

where ϕ is defined by

$$z' = \frac{\Delta}{\Delta^2 - 3\ell^{-2}}\phi\,. \tag{3.67}$$

In the final case it is also useful to record

$$X^1 = \frac{1}{4}\left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right) d\left[\rho^2\left(d\phi + \cos\theta d\psi\right)\right]. \tag{3.68}$$

In each case, the base space has a metric of cohomogeneity one, with a curvature singularity at $\rho = 0$. This is a good illustration of the point that a nakedly singular base space can correspond to a non-singular five-dimensional spacetime. Note that the metric (3.65) has a regular limit as $\Delta \to 0$ (define $z' = \Delta z''$) but this does *not* correspond to setting $\Delta = 0$ in the solution (3.50) (since that solution would be null). The metric (3.66) reduces to flat space in the limit $\Delta \to \infty$.

4. Supersymmetric black holes

4.1 Derivation of the solution

We have determined four possibilities for the near-horizon geometry of a supersymmetric solution with a horizon. It is natural to ask whether any of these can arise as the near-horizon geometry of a supersymmetric black hole. The solutions (3.45), (3.50) and (3.59) seem unlikely to arise from a solution that is asymptotically AdS (in the global sense) since their horizons are most naturally interpreted as spatially non-compact. However, the solution (3.57) naturally has a horizon of S^3 topology so it is natural to ask whether there is a corresponding black hole solution. In this section we shall obtain such solutions explicitly.

We shall start by deducing some properties of the base space associated with such a solution. Since the solution should be asymptotically AdS_5, we shall demand that the base space should asymptotically approach the base space of the AdS_5 solution. In [30] it was shown that the AdS_5 solution can be obtained by talking the base space to be the

Bergmann manifold. However, AdS_5 is maximally supersymmetric and it might therefore be possible to write the metric in the form (2.12) in more than one way (for example this happens for maximally supersymmetric solutions of the ungauged theory [27]). In the appendix, we show that this is not the case: the only way of writing AdS_5 as a timelike solution is to use the Bergmann manifold as the base space. The metric of the Bergmann manifold is

$$ds_4^2 = d\rho^2 + \frac{\ell^2}{4}\sinh^2\left(\frac{\rho}{\ell}\right)\left[(\sigma_L^1)^2 + (\sigma_L^2)^2\right] + \frac{\ell^2}{4}\sinh^2\left(\frac{\rho}{\ell}\right)\cosh^2\left(\frac{\rho}{\ell}\right)(\sigma_L^3)^2\,, \tag{4.1}$$

where σ_L^i are right-invariant 1-forms on SU(2). These can be expressed in terms of Euler angles (θ, ψ, ϕ) as

$$\begin{aligned}\sigma_L^1 &= \sin\phi d\theta - \cos\phi \sin\theta d\psi \\ \sigma_L^2 &= \cos\phi d\theta + \sin\phi\sin\theta d\psi \\ \sigma_L^3 &= d\phi + \cos\theta d\psi\,.\end{aligned} \tag{4.2}$$

where SU(2) is parametrized by taking $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 4\pi$ and $0 \leq \psi \leq 2\pi$. The right-invariant 1-forms obey

$$d\sigma_L^i = -\frac{1}{2}\epsilon_{ijk}\sigma_L^j \wedge \sigma_L^k\,. \tag{4.3}$$

The Kähler form is

$$X^1 = \frac{\ell^2}{4} d\left[\sinh\left(\frac{\rho}{\ell}\right)\sigma_L^3\right]. \tag{4.4}$$

Next we demand that near the horizon, the base space should approach the base space of the solution (3.57), i.e., it should agree with equation (3.66) at small ρ. Now (3.66) can be rewritten as

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}\left[\left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right)\left((\sigma_L^1)^2 + (\sigma_L^2)^2\right) + \left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right)^2 (\sigma_L^3)^2\right], \tag{4.5}$$

with Kähler form

$$X^1 = \frac{1}{4}\left(\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}\right) d\left[\rho^2\sigma_L^3\right]. \tag{4.6}$$

In summary, we need a base space that has an asymptotic region in which it resembles the large ρ behaviour of (4.1) and another region in which it resembles the (singular) small ρ behaviour of (4.5). This suggests that try the following cohomogeneity one ansatz for the base space of a black hole solution

$$ds_4^2 = d\rho^2 + a(\rho)^2\left((\sigma_L^1)^2 + (\sigma_L^2)^2\right) + b(\rho)^2(\sigma_L^3)^2\,, \tag{4.7}$$

with Kähler form

$$X^1 = d\left[c(\rho)\sigma_L^3\right]. \tag{4.8}$$

The surfaces of constant ρ are homogeneous, with a transitively acting $U(1)_L \times SU(2)_R$ isometry group, The $U(1)_L$ generated by $\partial/\partial\phi$ is manifestly a symmetry and the $SU(2)_R$ is a symmetry because σ_L^i is invariant under the right action of SU(2).

We shall assume $a, b > 0$ and introduce an orthonormal basis

$$e^0 = d\rho,\ e^1 = a\sigma^1,\ e^2 = a\sigma^2,\ e^3 = b\sigma^3 \tag{4.9}$$

with volume form $e^0 \wedge e^1 \wedge e^2 \wedge e^3$. The requirement that X^1 be an anti-self dual complex structure reduces to

$$c = -\epsilon a^2, \qquad b = 2aa'\,, \tag{4.10}$$

where $\epsilon = \pm 1$. We then have

$$X^1 = -\epsilon\left(e^0 \wedge e^3 - e^1 \wedge e^2\right). \tag{4.11}$$

The base space is now determined up to one arbitrary function, namely $a(\rho)$. The boundary conditions deduced above imply that we need a to be proportional to ρ as $\rho \to 0$ and proportional to $\exp(\rho/\ell)$ as $\rho \to \infty$.

We can write down the solution for f using equation (2.16):

$$f^{-1} = \frac{\ell^2}{12a^2a'}(4(a')^3 + 7aa'a'' - a' + a^2a''')\,. \tag{4.12}$$

Next we have to find a 1-form ω on $\mathcal{B}$ obeying equations (2.17) and (2.19). Once again, we shall obtain a natural ansatz for ω by examining AdS_5 and the near-horizon geometry. The AdS_5 solution has [30]

$$\omega = \frac{\ell}{2}\sinh^2\left(\frac{\rho}{\ell}\right)\sigma_L^3\,. \tag{4.13}$$

(At first it is a little surprising that a static solution such as AdS_5 should have non-zero ω but it turns out that the construction of [30] yields AdS_5 in non-static coordinates.) The near-horizon solution (3.57) can be written in the form (2.12) if we define

$$t = u + \frac{1}{(\Delta^2 + 9\ell^{-2})\,r}\,, \tag{4.14}$$

so $V = \partial/\partial t = \partial/\partial u$, and

$$\omega = \frac{3\sigma_L^3}{\Delta\ell\,(\Delta^2 - 3\ell^{-2})\,r} = \frac{12\sigma_L^3}{\ell\,(\Delta^2 - 3\ell^{-2})\,(\Delta^2 + 9\ell^{-2})\,\rho^2}\,. \tag{4.15}$$

If a supersymmetric black hole solution exists then it should have ω that behaves as (4.13) for large ρ and as (4.15) for small ρ. This suggests the Ansatz

$$\omega = \Psi(\rho)\sigma_L^3\,, \tag{4.16}$$

which gives

$$G^{\pm} = \frac{f}{2}\left(\frac{\Psi'}{2aa'} \mp \frac{\Psi}{a^2}\right)\left(e^0 \wedge e^3 \pm e^1 \wedge e^2\right). \tag{4.17}$$

It remains to solve equations (2.17) and (2.19). Equation (2.17) reduces to

$$\frac{\Psi'}{2aa'} - \frac{\Psi}{a^2} = \frac{\epsilon\ell g}{2f} \tag{4.18}$$

where

$$g = -\frac{a'''}{a'} - 3\frac{a''}{a} - \frac{1}{a^2} + 4\frac{a'^2}{a^2}\,. \tag{4.19}$$

Using equation (4.18) we can write (2.19) as

$$\frac{\Psi'}{2aa'} + \frac{\Psi}{a^2} = -\frac{\epsilon\ell}{2}\left(\nabla^2 f^{-1} + 8\ell^{-2} f^{-2} - \frac{\ell^2 g^2}{18}\right). \tag{4.20}$$

Note that

$$\nabla^2 f^{-1} = \frac{1}{a^3 a'}\partial_\rho\left(a^3 a' \partial_\rho f^{-1}\right). \tag{4.21}$$

Eliminating Ψ' between equations (4.18) and (4.20) gives

$$\Psi = -\frac{\epsilon\ell a^2}{4}\left(\nabla^2 f^{-1} + 8\ell^{-2} f^{-2} - \frac{\ell^2 g^2}{18} + f^{-1} g\right). \tag{4.22}$$

This expression uniquely determines Ψ in terms of a. Substituting this back into equation (4.18) or (4.20) then gives

$$\left(\nabla^2 f^{-1} + 8\ell^{-2} f^{-2} - \frac{\ell^2 g^2}{18} + f^{-1} g\right)' + \frac{4a'g}{af} = 0\,, \tag{4.23}$$

which is a sixth order nonlinear ordinary differential equation in $a(\rho)$. Everything else is determined once we have solved this equation. We are already familiar with two solutions of this equation, namely those corresponding to the AdS_5 and near-horizon geometries. Now we are looking for a more general solution that interpolates between these two types of behaviour. First we need to understand the boundary conditions at $\rho = 0$. From the near horizon geometry, we know that the coordinate transformation relating ρ to r must be of the form $r \propto \rho^2$ for small ρ. Hence smoothness of the near-horizon geometry implies that only even powers of ρ should appear in the five dimensional metric. Looking at the expression for f, this appears to imply that a must contain either only odd powers or only even powers of ρ. Since we need $a \propto \rho$ for small ρ we must take the former possibility. Hence we demand $a(0) = a''(0) = a^{(4)}(0) = 0$. The equation (4.23) determines the sixth derivative of a in terms of lower derivatives. Therefore demanding that $a^{(6)}(0) = 0$ gives an equation relating $a^{(5)}(0)$ to $a'''(0)$ and $a'(0)$. Hence the solutions of interest are determined by specifying $a'''(0)$ and $a'(0)$.

In fact by rescaling the coordinates $t = \lambda\tilde{t}$, $\rho = \tilde{\rho}/\sqrt{\lambda}$, $a(\rho)$ is replaced by $\tilde{a}(\tilde{\rho}) = \sqrt{\lambda}a(\tilde{\rho}/\sqrt{\lambda})$. This has the effect of rescaling $a'''(0)$ by λ^{-1} but leaves $a'(0)$ invariant. Hence $a'''(0)$ can be rescaled to any convenient value; it is only the *sign* of $a'''(0)$ that is important. Hence we would naively expect to find just three 1-parameter familes of solutions satisfying the boundary conditions required for a smooth horizon. The three families will correspond to $a'''(0)$ positive, negative or zero.

We are already familiar with a 1-parameter family of solutions with $a'''(0) = 0$: these are just the near-horizon solutions (4.5) with $a(\rho) \propto \rho$. They are labelled by the parameter Δ.

Another solution we know is the AdS_5 solution, with $a = (\ell/2)\sinh(\rho/\ell)$. This has $a'''(0) > 0$. The form of this solution suggests that we look for a more general solution of the form $a = \alpha\ell\sinh(\beta\rho/\ell)$. Amazingly, this turns out to be a solution to (4.23) for arbitrary α and β. It is really just a 1-parameter family of solutions since the above rescaling can be used to set β to any convenient value. We shall choose $\beta = 1$, i.e.,

$$a = \alpha\ell\sinh\left(\frac{\rho}{\ell}\right). \tag{4.24}$$

This is a 1-parameter family of solutions with $a'''(0) > 0$.

Finally, if we take this solution and analytically continue $\alpha \to -i\alpha$ and $\beta \to i\beta$ we obtain the solution $a = \alpha\ell\sin(\beta\rho/\ell)$. Again, we can rescale to set $\beta = 1$, i.e.,

$$a = \alpha\ell\sin\left(\frac{\rho}{\ell}\right). \tag{4.25}$$

This is a 1-parameter solution with $a'''(0) < 0$.

Of these three families of solutions, which is most likely to describe a black hole? By construction, all have a regular horizon at $\rho = 0$. However, to describe a black hole they must also be asymptotically AdS_5. The first family just describes the near-horizon geometry and does not have the correct asymptotics. The third family looks as if it has problems at $\rho = \ell\pi/2$ ($b = 0$ there). This leaves the second family. We shall now show that this does indeed describe a 1-parameter family of supersymmetric black holes in AdS_5.

First, comparing the small ρ behaviour of (4.24) with the corresponding behaviour in (4.5) we find that our solution has a regular near-horizon geometry with parameter Δ determined by

$$\alpha = \frac{1}{2}\sqrt{\frac{\Delta^2 + 9\ell^{-2}}{\Delta^2 - 3\ell^{-2}}}, \tag{4.26}$$

hence $\alpha > 1/2$ is necessary for the solution to have a regular horizon. $\alpha = 1/2$ does not give a horizon: it is just the AdS_5 solution in global coordinates.

Equations (4.10), (4.12) and (4.22) give the full solution as

$$a = \alpha\ell\sinh\left(\frac{\rho}{\ell}\right), \qquad b = 2\alpha^2\ell\sinh\left(\frac{\rho}{\ell}\right)\cosh\left(\frac{\rho}{\ell}\right), \qquad c = -\epsilon\alpha^2\ell^2\sinh^2\left(\frac{\rho}{\ell}\right), \tag{4.27}$$

$$f^{-1} = 1 + \frac{4\alpha^2 - 1}{12\alpha^2\sinh^2(\rho/\ell)}, \tag{4.28}$$

$$\Psi = -2\epsilon\alpha^2\ell\sinh^2\left(\frac{\rho}{\ell}\right)\left[1 + \frac{4\alpha^2 - 1}{4\alpha^2\sinh^2(\rho/\ell)} + \frac{(4\alpha^2 - 1)^2}{96\alpha^4\sinh^4(\rho/\ell)}\right]. \tag{4.29}$$

The Maxwell field strength is obtained from equation (2.14). It can be written in terms of the potential

$$A = \frac{\sqrt{3}}{2}f\left[dt + \frac{\epsilon\ell(4\alpha^2 - 1)^2}{144\alpha^2\sinh^2(\rho/\ell)}\sigma_L^3\right]. \tag{4.30}$$

4.2 New coordinates

It is convenient to introduce a new radial coordinates R defined by

$$\frac{1}{4}R^2 = f^{-1}a^2, \tag{4.31}$$

that is

$$R = \ell\sqrt{4\alpha^2 \sinh^2\left(\frac{\rho}{\ell}\right) + \frac{4\alpha^2 - 1}{3}}\,. \tag{4.32}$$

Let

$$R_0 = \ell\sqrt{\frac{4\alpha^2 - 1}{3}}\,. \tag{4.33}$$

We then have (recall that $\epsilon = \pm 1$)

$$f = 1 - \frac{R_0^2}{R^2}\,, \qquad \Psi = -\frac{\epsilon R^2}{2\ell}\left(1 + \frac{2R_0^2}{R^2} + \frac{3R_0^4}{2R^2(R^2 - R_0^2)}\right). \tag{4.34}$$

The horizon is at $R = R_0$. The full metric is

$$ds^2 = -f^2 dt^2 - 2f^2\Psi dt\sigma_L^3 + U(R)^{-1}dR^2 + \frac{R^2}{4}\left[(\sigma_L^1)^2 + (\sigma_L^2)^2 + \Lambda(R)(\sigma_L^3)^2\right], \tag{4.35}$$

where σ_L^i was defined in equation (4.2), and

$$U(R) = \left(1 - \frac{R_0^2}{R^2}\right)^2\left(1 + \frac{2R_0^2}{\ell^2} + \frac{R^2}{\ell^2}\right), \qquad \Lambda(R) = 1 + \frac{R_0^6}{\ell^2 R^4} - \frac{R_0^8}{4\ell^2 R^6}\,. \tag{4.36}$$

The Maxwell potential is

$$A = \frac{\sqrt{3}}{2}\left[\left(1 - \frac{R_0^2}{R^2}\right)dt + \frac{\epsilon R_0^4}{4\ell R^2}\sigma_L^3\right]. \tag{4.37}$$

One final coordinate transformation is required to demonstrate that the solution is asymptotically AdS_5. Let

$$\phi' = \phi + \frac{2\epsilon t}{\ell} \tag{4.38}$$

and

$$\Omega(R) = \frac{2\epsilon}{\ell\Lambda(R)}\left[\left(\frac{3}{2} + \frac{R_0^2}{\ell^2}\right)\frac{R_0^4}{R^4} - \left(\frac{1}{2} + \frac{R_0^2}{4\ell^2}\right)\frac{R_0^6}{R^6}\right]. \tag{4.39}$$

The metric becomes

$$ds^2 = -U(R)\Lambda(R)^{-1}dt^2 + U(R)^{-1}dR^2 + \frac{R^2}{4}\left((\sigma_L^{1'})^2 + (\sigma_L^{2'})^2 + \Lambda(R)\left(\sigma_L^{3'} - \Omega(R)dt\right)^2\right), \tag{4.40}$$

where $\sigma_L^{i'}$ is defined in the same way as σ_L^i (equation (4.2)) but with ϕ replaced by ϕ'. The electromagnetic potential is

$$A = \frac{\sqrt{3}}{2}\left[\left(1 - \frac{R_0^2}{R^2} - \frac{R_0^4}{2\ell^2 R^2}\right)dt + \frac{\epsilon R_0^4}{4\ell R^2}\sigma_L^{3'}\right]. \tag{4.41}$$

Now $\Lambda \to 1$, $\Omega \to 0$ and $U \sim R^2/\ell^2$ as $R \to \infty$ so the solution is manifestly asymptotic to AdS_5. By construction it has a regular horizon at $R = R_0$. It is well-behaved for $R > R_0$. Hence it describes a 1-parameter family of supersymmetric, asymptotically AdS_5 black holes. The isometry group is $R \times \mathrm{U}(1)_L \times \mathrm{SU}(2)_R$, the same as that of the non-static supersymmetric black holes of the ungauged theory [35].

4.3 Properties of the solution

We have seen that our solution is asymptotically AdS_5. It therefore has the Einstein universe $R \times S^3$ as its conformal boundary. In the coordinates induced from our bulk solution (4.40), the boundary metric can be written

$$ds^2 = -dt^2 + \frac{\ell^2}{4}\left((\sigma_L^{1'})^2 + (\sigma_L^{2'})^2 + (\sigma_L^{3'})^2\right). \tag{4.42}$$

The S^3 has radius ℓ. Boundary time translations are generated by $\partial/\partial t$ so it is natural to use $\partial/\partial t$ as the generator of bulk time translations too. However, for rotating, asymptotically AdS, black holes, there is a subtlety [39]: there exists another timelike Killing vector field in the bulk. In our case, this is the Killing vector field V associated with supersymmetry. In the coordinates of (4.40), this is given by

$$V = \frac{\partial}{\partial t} + \frac{2\epsilon}{\ell}\frac{\partial}{\partial \phi'}. \tag{4.43}$$

Furthermore, V is tangent to the event horizon of the black hole so using V as the generator of time translations corresponds to working in a co-rotating frame. The advantage of using V as the generator of time translations is that V is timelike everywhere outside the black hole, whereas $\partial/\partial t$ becomes spacelike near the event horizon. Hence if $\partial/\partial t$ generates time translations then there is an ergoregion, but if the co-rotating Killing vector field V generates time translations then there is no ergoregion. This is a general feature of rotating, asymptotically AdS, black holes [40].

Note that V is null on the conformal boundary with metric (4.42). In this sense, the boundary rotates at the speed of light in the co-rotating frame. The same is true of supersymmetric rotating black holes in $D = 3$ [39] and $D = 4$ [20]. Conversely, if we take $\partial/\partial t$ as the generator of time translations then equation 4.43 shows that the angular velocity of the black hole in the ϕ' direction is

$$\Omega_H = \frac{2\epsilon}{\ell}, \tag{4.44}$$

which implies that the black hole rotates at the speed of light with respect to the frame at infinity.

We are ultimately interested in the AdS/CFT description of these black holes. From a CFT perspective, it seems most natural to use $\partial/\partial t$ (in the coordinates (4.40)) as the generator of time translations so we shall do that henceforth.

Ashtekar and Das (AD) have shown how to define conserved quantities in asymptotically anti-de Sitter spacetimes of arbitrary dimension $D \geq 4$ [41]. There is a conserved quantity associated with each symmetry of the conformal boundary. The AD mass is the conserved quantity associated with $\partial/\partial t$ and takes the value

$$M = \frac{3\pi R_0^2}{4G}\left(1 + \frac{3R_0^2}{2\ell^2} + \frac{2R_0^4}{3\ell^4}\right). \tag{4.45}$$

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We define the angular momentum to be *minus*[5] the conserved quantity associated with $\partial/\partial\phi'$, which gives

$$J = \frac{3\epsilon\pi R_0^4}{8G\ell}\left(1+\frac{2R_0^2}{3\ell^2}\right), \tag{4.46}$$

and the angular momentum associated with $\partial/\partial\psi$ vanishes. Hence our solutions carry equal angular momenta in two orthogonal 2-planes, just like the supersymmetric black holes of the ungauged theory [5, 35]. Note that the choice of the sign ϵ fixes the sign of J.

An alternative definition of conserved charges is provided by the "holographic stress tensor" approach [42, 43, 44, 45]. In this method, the expectation value of the stress tensor of the dual CFT on $R \times S^3$, in a conformal frame with metric (4.42), is calculated using the formula

$$\langle T_{\mu\nu}\rangle = \lim_{R\to\infty}\frac{R^2}{8\pi G\ell^2}\left[-\left(K_{\mu\nu}-Kh_{\mu\nu}\right)-\frac{3}{\ell}h_{\mu\nu}+\frac{\ell}{2}\left(\bar{R}_{\mu\nu}-\frac{1}{2}\bar{R}h_{\mu\nu}\right)\right]. \tag{4.47}$$

The right hand side is defined in terms of tensors associated with a surface $R =$ constant in the coordinates of (4.40). This surface has induced metric $h_{\mu\nu}$, extrinsic curvature $K_{\mu\nu}$ with trace K, Ricci tensor $\bar{R}_{\mu\nu}$ and Ricci scalar $\bar{R}$. An overall factor of R^2/ℓ^2 arises from the conformal transformation required to map $h_{\mu\nu}$ to the metric (4.42) as $R\to\infty$. We obtain

$$\begin{aligned}8\pi G\langle T_{\mu\nu}\rangle dx^\mu dx^\nu = {} & \frac{1}{\ell}\left(\frac{3}{8}+\frac{3R_0^2}{\ell^2}+\frac{9R_0^4}{2\ell^4}+\frac{2R_0^6}{\ell^6}\right)dt^2-2\epsilon\left(\frac{3R_0^4}{2\ell^4}+\frac{R_0^6}{\ell^6}\right)dt\sigma_L^{3'}+\\ & +\frac{\ell}{32}\left(1+\frac{8R_0^2}{\ell^2}+\frac{12R_0^4}{\ell^4}\right)\left[(\sigma_L^{1'})^2+(\sigma_L^{2'})^2\right]+\\ & +\frac{\ell}{32}\left(1+\frac{8R_0^2}{\ell^2}+\frac{12R_0^4}{\ell^4}+\frac{16R_0^6}{\ell^6}\right)(\sigma_L^{3'})^2 .\end{aligned} \tag{4.48}$$

The total energy E is obtained by integrating $\langle T_{tt}\rangle$ over S^3, giving

$$E = M + \frac{3\pi\ell^2}{32G}. \tag{4.49}$$

The final term is just the Casimir energy of the CFT on $R\times S^3$, which is interpreted as the energy of global AdS_5 in this approach [42]. The total angular momentum is obtained by integrating $-\langle T_{t\phi'}\rangle$ over S^3, which gives the same result J as the AD method used above.

Define the electric charge by

$$Q = \frac{1}{4\pi G}\int_{S^3_\infty}\star F, \tag{4.50}$$

where the integral is over the three-sphere at infinity on a surface of constant t. The orientation of spacetime can be deduced from the orientation of the base space to be

[5]The minus sign is required to give the correct sign for the angular momentum of known black hole solutions, such as those of [39]. It is the same relative minus sign as occurs in the definition of conserved quantities for geodesics: if U is the tangent vector to a geodesic then the energy is $E=-U\cdot\partial/\partial t$ and the angular momentum is $L=+U\cdot\partial/\partial\phi$.

$dt \wedge dR \wedge \sigma_L^1 \wedge \sigma_L^2 \wedge \sigma_L^3$. For our solution we find[6]

$$Q = \frac{\sqrt{3}\pi R_0^2}{2G}\left(1 + \frac{R_0^2}{2\ell^2}\right). \tag{4.51}$$

We then have

$$M - \frac{2}{\ell}|J| = \frac{\sqrt{3}}{2}|Q|\,. \tag{4.52}$$

Note that the left hand side is the AD conserved quantity associated with V. The BPS bound for this theory is [15][7]

$$M - \frac{|J_1| + |J_2|}{\ell} \geq \frac{\sqrt{3}}{2}|Q| \tag{4.53}$$

where $J_{1,2}$ are the angular momenta associated with the orthogonal 2-planes at infinity. Our black hole saturates this inequality with $J_1 = J_2 = J$. Although we derived our solution within the framework of [30], the quickest way of checking it is to verify that it satifies the equations of motion. It must then be supersymmetric because it saturates the BPS bound.

Our solution has a regular horizon by construction. To find out what lies behind the horizon, we transform to gaussian null coordinates as follows:

$$dt = du - \Lambda U^{-1} dr\,, \qquad d\phi = d\phi'' - \frac{4f^2\Psi}{R^2 U} dr\,, \qquad dR = \Lambda^{1/2} dr\,, \tag{4.54}$$

with $R = R_0$ at $r = 0$. In these coordinates, the metric is

$$ds^2 = -f^2 du^2 + 2du dr - 2f^2 \Psi du \sigma_L^{3''} + \frac{R^2}{4}\left[(\sigma_L^{1''})^2 + (\sigma_L^{2''})^2 + \Lambda(\sigma_L^{3''})^2\right], \tag{4.55}$$

where we have used $UR^2 = f^2(4f^2\Psi^2 + R^2\Lambda)$, and $\sigma_L^{i''}$ is defined in the same way as σ_L^i (equation (4.2)) but with ϕ replaced by ϕ''. Note that Λ approaches a non-zero constant at $R = R_0$ so $R - R_0$ is proportional to r there. It is then easy to see that $f^2 \propto r^2$ and $f^2\Psi \propto r$ as $r \to 0$, so the above metric is of the form (3.1), with a smooth event horizon at $r = 0$. A gauge transformation $A \to A + d\lambda$ with $\lambda = \lambda(r)$ makes A regular at the horizon with A a constant multiple of $\sigma_L^{3''}$ on the horizon. It is straightforward to read off the spatial geometry of the horizon:

$$ds_3^2 = \frac{R_0^2}{4}\left[(\sigma_L^{1''})^2 + (\sigma_L^{2''})^2 + \left(1 + \frac{3R_0^2}{4\ell^2}\right)(\sigma_L^{3''})^2\right], \tag{4.56}$$

a squashed S^3. This is no surprise because the black hole was constructed to have near-horizon geometry (3.57). The area of the horizon is

$$\mathcal{A} = 2\pi^2 R_0^3 \sqrt{1 + \frac{3R_0^2}{4\ell^2}}\,. \tag{4.57}$$

[6]One might wonder why this is always non-negative. This is related to our choice $f > 0$. Solutions with $f < 0$ are related by a change of orientation [27], and will have the opposite sign for Q.

[7]The final result presented in [15] assumed $J_1 = J_2 = 0$ but it is straightforward to use the intermediate results of [15] to take account of non-zero J_i.

In the region $r < 0$ behind the horizon, we can invert the above coordinate transformation to return to the original coordinates $(t, R, \theta, \psi, \phi)$. The metric is just (4.35) with $R < R_0$. This metric is smooth for $R > 0$ but has a curvature singularity at $R = 0$. Note that there is a value $R = R_*$, $0 < R_* < R_0$ with $\Lambda(R_*) = 0$. Hence $\partial/\partial\phi$ is null at $R = R_*$. However, ϕ is periodic hence there are closed null curves. Similarly, for $R < R_*$ there are closed timelike curves. Hence the interior of the black hole contains a curvature singularity surrounded by a region of closed timelike curves.

In the gaussian null coordinates, a curve of constant u, θ, ψ and ϕ'' with $r = -\lambda$ is a future-directed null geodesic with affine parameter λ. Let $r_* < 0$ denote the value of r at which $R = R_*$. It is easy to see that

$$R - R_* \approx \frac{1}{4}\Lambda'(R_*)(r - r_*)^2 . \tag{4.58}$$

Hence, as r decreases through r_*, R decreases to R_* but then starts to increase again. As r decreases further, R will increase through R_0 so this null geodesic will eventually emerge into another asymptotically AdS region.

These considerations suggest that the global causual structure of our solution is very similar to that of the supersymmetric rotating black holes of the ungauged theory, which was explored in detail in [36]. The only qualitative difference is that our solution is asymptotically AdS rather than asymptotically flat.

Finally, we note that if we take the limit $\ell \to \infty$ with R_0 held fixed then our solution reduces to a *static* supersymmetric black hole solution of the ungauged supergravity theory. The four dimensional solutions of [19] behave similarly.

4.4 A limiting case

We shall now examine the geometry of our solution when the black hole is very large, i.e., $R_0 \to \infty$. To this end, it is convenient to write σ^3_L as

$$\sigma^3_L = (d\phi + d\psi) - 2\sin^2\frac{\theta}{2}\, d\psi , \tag{4.59}$$

and define new coordinates S, $\tilde{\theta}$, z

$$S = \frac{R}{R_0}, \qquad \tilde{\theta} = \frac{1}{2}R_0\theta , \qquad z = \frac{\sqrt{3}R_0^2}{4\ell}(\phi + \psi) . \tag{4.60}$$

We then take the limit $R_0 \to \infty$ holding t, S, $\tilde{\theta}$, z and ψ fixed. If we define

$$x = \tilde{\theta}\cos\psi , \qquad y = \tilde{\theta}\sin\psi , \tag{4.61}$$

then the limiting form of the solution is

$$\begin{aligned} ds^2 &= -\left(1 - S^{-2}\right)^2 \left[dt^2 - \frac{4\epsilon}{\sqrt{3}}\left(S^2 + 2 + \frac{3}{2(S^2-1)}\right) dt \left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)\right] + \\ &\quad + \frac{\ell^2 dS^2}{(1-S^{-2})^2(2+S^2)} + S^2(dx^2 + dy^2) + \frac{1}{3}(4S^{-2} - S^{-4})\left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)^2, \\ A &= \frac{\sqrt{3}}{2}\left[\left(1 - S^{-2}\right) dt + \frac{\epsilon}{\sqrt{3}S^2}\left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)\right] . \end{aligned} \tag{4.62}$$

By transforming to gaussian null coordinates in the manner described above, it is easy to see that this solution has a regular horizon at $S = 1$, with near-horizon geometry (3.45). Thus taking this limit has the effect of performing a group contraction of H from S^3 to Nil.

This solution is not asymptotically AdS: as $S \to \infty$, it tends to the following vacuum solution

$$\begin{aligned} ds^2 &= -\left[dt - \frac{2\epsilon S^2}{\sqrt{3}}\left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)\right]^2 + \\ &\quad + \frac{\ell^2 dS^2}{S^2} + S^2\left(dx^2 + dy^2\right) + \frac{4S^4}{3}\left(dz + \frac{\sqrt{3}}{2\ell}(ydx - xdy)\right)^2, \\ F &= 0\,. \end{aligned} \tag{4.63}$$

It can be verified that this describes a supersymmetric solution in its own right. Note that, although this solution belongs to the timelike family, $\partial/\partial z$ is a null Killing vector field. Null geodesics tangent to $\partial/\partial z$ are free of expansion, rotation and shear so this solution is a plane-fronted wave.

5. Discussion

It is natural to ask whether our 1-parameter family of solutions is a special example in some larger family of supersymmetric, asymptotically AdS_5 black hole solutions. There are two reasons why this seems unlikely. First, if there was a more general family of solutions then presumably there would be a more general family of near-horizon solutions than 3.57. This more general family would have to have non-constant Δ on the horizon. We are not aware of any black hole solution in any dimension for which Δ is non-constant. Secondly, the results of [19] show that in $D = 4$ there is at most one supersymmetric black hole for given charge(s). Our $D = 5$ solution has the same property, but this would not be true if it were part of some larger family.

Of course, we would expect more general solutions to exist in more complicated theories. Of particular interest is the maximal $\mathcal{N} = 4$ $D = 5$ gauged supergravity (i.e., 32 supercharges) with SO(6) gauge group arising from Kaluza-Klein reduction of type-IIB supergravity on S^5. This can be truncated to a $\mathcal{N} = 1$ theory with $U(1)^3$ gauge symmetry. When embedded in this theory, our solutions have equal U(1) charges but it should be straightforward to generalize them to obtain solutions with three independent charges. The form of our solution suggests a natural Ansatz for the metric and gauge fields of such solutions, and one would assume the scalar fields to be functions of R alone. It would also be interesting to find non-extremal generalizations of our solution. These should form a 4-parameter family of solutions of minimal $D = 5$ gauged supergravity, parametrized by the mass, charge and two independent angular momenta. They should reduce to the solutions of [39] in the limit of vanishing charge.

Another interesting question concerns black hole uniqueness. General stationary asymptotically flat $D = 5$ black holes do not obey a uniqueness theorem [46]. However, supersymmetric black holes do [33], at least in minimal supergravity. It is not known whether

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similar behaviour occurs for asymptotically AdS black holes. It seems unlikely that the proof of [33] could be extended to gauged supergravity. The first problem is in classifying all near-horizon geometries. We were unable to do this without additional assumptions. However, it might be possible to perform a complete analysis. The second problem is more serious, namely that the proof relies on a global property of hyper-Kähler manifolds that has no Kähler analogue.

The near-horizon solutions that we found preserve at least 1/4 supersymmetry. In the ungauged theory, the near-horizon geometry of a supersymmetric black hole is actually maximally supersymmetric [35]. This cannot be true in the gauged theory because the only maximally supersymmetric solution is AdS_5 [30]. However, the near-horizon solutions might be 1/2 or 3/4 supersymmetric, as seems to be the case with the $AdS_3 \times H^2$ solution. It might be possible to extend the work of [30] to classify all 1/2 and 3/4 supersymmetric solutions of this theory.

The main objective of our work was to stimulate further investigation of black holes using the AdS/CFT correspondence. Supersymmetric, asymptotically AdS, black holes have been studied as limits of non-supersymmetric solutions in $D = 3$ [39] and $D = 4$ [20]. The matter of the dual CFT rotates around the Einstein universe with a speed that approaches the speed of light as the supersymmetric limit is taken. It seems very likely that the same will be true of the CFT states corresponding to our solution. (As we have seen, the boundary rotates at the speed of light with respect to the co-rotating frame in the bulk.) We hope that by counting such states (with appropriate R-charge), it will be possible to reproduce the entropy of our solutions.

It would also be interesting to obtain a microscopic understanding of these black holes from a bulk perspective. Our solution can be embedded into type-IIB supergravity using [21]. The only non-trivial fields are the metric and 5-form, which suggests that the bulk description should only involve D3-branes. The charge of our solutions is Kaluza-Klein charge arising from momentum on the internal S^5. BPS solutions describing probe D3-branes moving on the internal S^5 of $AdS_5 \times S^5$ are known: giant gravitons [47]. Going beyond the probe approximation, one can construct supersymmetric solutions of minimal $D = 5$ gauged supergravity by considering large numbers of giant gravitons suitably distributed on S^5 [48]. These solutions correspond to static point sources in AdS_5, and are therefore nakedly singular. However, as we have shown, the inclusion of angular momentum can lead to a regular horizon. Giant gravitons that carry angular momentum both on S^5 and in AdS_5 were investigated in [49]. Perhaps our solutions can be interpreted as a distribution of such objects.

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A. Maximally symmetric solutions

In [30] it was shown that by taking the Kähler base space with a Bergmann metric, one can obtain the maximally symmetric AdS_5 geometry in five dimensions. Here we shall prove a (partial) converse, namely that the only maximally symmetric timelike solution of five dimensional minimal gauged supergravity is AdS_5 (with AdS radius ℓ), and that the base space is locally isometric to the Bergmann manifold.

To show this, suppose that

$$ {}^5R_{\alpha\beta\rho\sigma} = \frac{\lambda}{4}(g_{\beta\rho}g_{\alpha\sigma} - g_{\alpha\rho}g_{\beta\sigma}) \tag{A.1} $$

for some constant λ. Hence, in particular, ${}^5R_{\alpha\beta} = -\lambda g_{\alpha\beta}$. So, the Einstein equations imply that

$$ F^2 = \frac{60}{\ell^2} - 15\lambda \tag{A.2} $$

and hence

$$ 3g_{\alpha\beta}(\frac{4}{\ell^2} - \lambda) = F_{\alpha\gamma}F_\beta{}^\gamma \tag{A.3} $$

If $\lambda \neq \frac{4}{\ell^2}$ then (A.3) implies that the determinant of the metric vanishes. To prevent this, we must have $\lambda = \frac{4}{\ell^2}$, so the five-dimensional geometry is AdS_5 with AdS radius ℓ. From $F_{\alpha\gamma}F_\beta{}^\gamma = 0$ we obtain $F = 0$.

Using the identity for F given in (2.19) we see that f must be constant. Without loss of generality we can set $f = 1$. In addition,

$$ \begin{aligned} G^+ &= 0 \\ d\omega &= \frac{2}{\ell}X^1 . \end{aligned} \tag{A.4} $$

Note that the only non-vanishing components of the spin connection are fixed by

$$ \begin{aligned} \omega_{0ij} &= \omega_{i0j} = \frac{1}{\ell}X^1{}_{ij} \\ \omega_{ijk} &= \hat{\omega}_{ijk} \end{aligned} \tag{A.5} $$

where here $\hat{\omega}_{ijk}$ denotes the spin connection of the Kähler base, and i, j, k are frame indices with respect to a vielbein on the base. Using these expressions it is straightforward to compute the following components of the Riemann tensor

$$ {}^5R_{ijpq} = {}^4R_{ijpq} + \frac{2}{\ell^2}\left(X^1{}_{ij}X^1{}_{pq} + \frac{1}{2}(X^1{}_{pi}X^1{}_{qj} - X^1{}_{qi}X^1{}_{pj})\right) . \tag{A.6} $$

However, from (A.1) we must have

$$ {}^5R_{ijpq} = \frac{1}{\ell^2}(\delta_{jp}\delta_{iq} - \delta_{ip}\delta_{jq}) \tag{A.7} $$

and so solving for ${}^4R_{ijpq}$ we obtain

$$ {}^4R_{ijpq} = \frac{2}{\ell^2}\left(-X^1{}_{ij}X^1{}_{pq} + \frac{1}{2}(X^1{}_{qi}X^1{}_{pj} - X^1{}_{pi}X^1{}_{qj}) - \frac{1}{2}(\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq})\right) \tag{A.8} $$

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This expression implies that the Kähler base space has constant holomorphic sectional curvature, which must be the same as that of the Bergmann manifold (since we know that is a solution to these equations). However, any two Kähler manifolds with the same constant holomorphic sectional curvature are locally holomorphically isometric [50]. Hence the base space must be locally holomorphically isometric to the Bergmann manifold.

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